M.Sc (I) Sem.

Compactness for Normed Spaces:

Definition:(1) Let X be a normed space and A be its arbitrary subset.

A. An element x of A is called its interior point if there is an open sphere $S_\epsilon(x)$, with centre x and radius ϵ, which is wholly contained in A. A is called an open set in X or an open subset of X if every point in it is interior point.

B. A is called a closed set in X if its complement is an open set or A is called a closed set if all convergent sequences in A have their limit points in A.

C. The closure of A, which is denoted by $\bar{A}$ is a closed set containing A which is formed by adding all limit points of sequences of A to A. A is closed iff $A = \bar{A}$.

D. A class (O_i) of open subsets of X is said to be an open cover of X if each point in X belongs to atleast one O_i i.e; if $X = \bigcup_i$. A subclass of an open cover, which is itself an open cover is called a subcover.

Definition:(2) A subset A of a normed space X is called compact if every open cover of it has a finite subcover.

Equivalently, a subset A of a normed space X is called compact if every sequence in A contains a convergent subsequence with a limit point in A.

Theorem:- Every compact subset of a normed space is bounded but the converse may be not true.

Proof:- We prove that first part by the method of contradiction. Suppose a compact subset A of a normed space x is not bounded. Since A is compact every open covering of A, consisting of open spheres with radius 1 and centres as each of its points, contains a finite subcovering.

$$A \subset \bigcup_{x \in A}^{n} S_1(x) \Rightarrow \text{there } ~~exists~~ \text{ must exist}$$

$x_1, x_2, \ldots x_n$ such that $A \subseteq \bigcup_{i=1} S_1(x_i)$ ~~implies that~~ ~~there must~~ suppose $m = \max_{1 \le i \le n} \|x_i\|$. Since A is not bounded, assume that there exists an element $x \in A$ such that $\|x\| > 1+m$. Since $x \in A$ and $A \subset \bigcup_{i=1}^{n} S_1(x_i)$, there must exist an element x_i such that $x \in S_1(x_i)$ this implies that $\|x - x_i\| < 1$. Applying the triangle inequality of the norm we find that

$$\|x\| = \|x - x_i + x_i\|$$
$$\le \|x - x_i\| + \|x_i\| \le 1 + \max\|x_i\| = 1+m$$

or $\|x\| \le 1+m$

This contradicts the fact that $\|x\| > 1+m$. Hence A is bounded.

For the converse, consider the normed space $C[0,\pi]$ and its subset

$$A = \{\sin nt\},\ n = 1, 2, \ldots \text{ and } t \in [0, \pi]$$

A is bounded but it is not compact.

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Theorem :— Every Compact subset of a normed Space is Complete.

Proof :— Let $\{x_n\}$ be a Cauchy sequence of a Compact subset Y of $(X, \|\cdot\|)$.

Since Y is Compact, $\{x_n\}$ Contains a Convergent subsequence

say $\{x_{nk}\} \to x_0 \in Y$.

For any k, we have $\|x_k - x_0\| \leq \|x_k - x_{nk}\| + \|x_{nk} - x_0\| \to 0$ as $k \to \infty$.

$(\|x_k - x_{nk}\| \to 0$ as $\{x_n\}$ is a Cauchy sequence

And $\|x_{nk} - x_0\| \to 0$ as $x_{nk} \to x_0$

Therefore, $\{x_n\}$ is Convergent and Consequently Y is complete.

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